JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 2, May 2017
http://dx.doi.org/10.14403/jcms.2017.30.2.177

CHAIN TRANSITIVE SETS AND DOMINATED SPLITTING FOR GENERIC DIFFEOMORPHISMS

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Abstract. Let $f : M \to M$ be a diffeomorphism of a compact smooth manifold M. In this paper, we show that C^1 generically, if a chain transitive set Λ is locally maximal then it admits a dominated splitting. Moreover, C^1 generically if a chain transitive set Λ of f is locally maximal then it has zero entropy.

1. Introduction

Let M be a closed C^∞ Riemannian manifold with $\dim M \geq 2$, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

Let Λ be a closed f invariant set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. In differentiable dynamical systems, the notion is an important concept. For that, many results published in [3, 6, 7, 8, 9, 10, 11, 12, 14]. In fact, they were used to various dynamical properties (expansive, continuum-wise expansive, continuum-wise fully expansive, shadowing, inverse shadowing, average shadowing, asymptotic average shadowing, etc). In the paper, we consider that if a closed invariant set which is locally maximal then it admits a dominated splitting for C^1 generic sense. An invariant closed set Λ is called *chain transitive* if for

Received November 04, 2016; Accepted February 01, 2017.
2010 Mathematics Subject Classification: Primary 37D30.
Key words and phrases: chain transitive, generic, locally maximal, dominated splitting, zero entropy.
This work is supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (No. 2014R1A1A1A05002124).

any $\delta > 0$ and $x, y \in \Lambda$, there is δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1) \subset \Lambda$ such that $x_0 = x$ and $x_n = y$.

We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Here the neighborhood U is called *locally maximal neighborhood* of Λ. We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property "P" is said to be *(C^1)-generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}(M)$. We use the terminology "for C^1 generic f" to express "there is a residual subset $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G} \ldots$". In the paper, we show the following which is a main theorem.

Theorem A *For C^1 generic $f \in \mathrm{Diff}(M)$, if a chain transitive set Λ of f is locally maximal then it admits a dominated splitting.*

2. Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. We say that $p \in P(f)$ with period $\pi(p)$ is a *sink* if all the eigenvalues of $D_p f^{\pi(p)}$ are less than 1, and $p \in P(f)$ with period $\pi(p)$ is a *source* if all eigenvalues of $D_p f^{\pi(p)}$ is greater than 1.

THEOREM 2.1. [1, Theorem 2.1] *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that given any chain transitive set Λ of $f \in \mathcal{G}$ then either there is a dominated splitting over Λ or the set Λ is contained in the Hausdorff limit of a sequence of periodic sinks or sources of f.*

We also recall that the Hausdoroff distance between two compact subsets A and B of M is given by:

$$d_H(A, B) = \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\}.$$

LEMMA 2.2. *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any chain transitive set Λ of $f \in \mathcal{G}$, if Λ is locally maximal and it does not admits a dominated splitting then Λ contains a sink or a source.*

Proof. Let $f \in \mathcal{G}$ and let U be a locally maximal neighborhood of Λ. Suppose that Λ does not admits a dominated splitting. Since Λ is compact, there is $\eta > 0$ such that $\Lambda \subset B_\eta(\Lambda) \subset U$. Since Λ does not admits a dominated splitting, by Theorem 2.1, there is a sequence of periodic sinks $Orb(s_n)$ such that $Orb(s_n)$ is the Hausdorff limit to Λ.

For sufficiently large n, we have $d_H(Orb(s_n), \Lambda) < \eta/2$. Then there is a periodic sink $s \in Orb(s_n)$ such that $s \in B_{\eta/2}(\Lambda) \subset U$. Since Λ is a locally maximal in U, we know that $s \in \Lambda$. The case of a sequence of periodic source is similar. □

Proof of Theorem A. Let $f \in \mathcal{G} \cap \mathcal{D}$. Assume that a locally maximal chain transitive set Λ does not admit a dominated splitting. Since Λ does not admits a dominated splitting, by Lemma 2.2 we know that Λ contains a sink or a source. Since Λ is a chain transitive set of f by [13, Lemma 2.1], Λ has neither sinks nor sources. This is a contradiction by Theorem 2.1. Thus C^1 generically, a chain transitive set Λ admits a dominated splitting if Λ is locally maximal. □

A compact f invariant set Δ is said to be *transitive* if there is a point $x \in \Delta$ such that $\omega(x) = \Delta$, where $\omega(x)$ is the omega limit set of x. In general, a chain transitive set is not a transitive set (see [4, Example 1.5]).

LEMMA 2.3. [4, Corollary 2] *There is a residual set $\mathcal{C} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{C}$, a chain transitive set Λ of f is a transitive set Δ of f.*

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. Let p be a hyperbolic periodic point of f. Then the homoclinic class of $Orb(p)$ is the set $H_f(p) = H_f(Orb(p)) = W^s(p) \pitchfork W^u(p)$, and a neighborhood V of $Orb(p)$, then the homoclinic class of p relative to V is the set

$$H_V(Orb(p)) = H_V(p) = \overline{\{x \in W^s(p) \pitchfork W^u(p) : Orb(x) \subset V\}}.$$

It is clear that if the homoclinic class $H_f(p)$ is locally maximal then it is relative to V, that is, $H_f(p) = H_V(p)$.

LEMMA 2.4. [2, Thereom 4.10] *There is a residual set $\mathcal{T} \subset \mathrm{Diff}(M)$ such that for any transitive set Λ of f if the transitive set Λ is locally maximal then $\Lambda = H_f(p)$ for some periodic point p of f.*

It is well known that if a diffeomorphism is More-Smale then it has zero entropy. The set of diffomorphisms having zero entropy is contained in the closure of the Morse-Smale diffeomorphism. Denote by $\mathcal{MS}$ the set of all Mores-Smale diffeomorphisms. $U = \mathrm{Diff}(M^2) \setminus \overline{\mathcal{MS}}$. Then

Pujals and Smbarino [15] proved that there exists an open and dense set $\mathcal{R} \subset U$ such that every $f \in \mathcal{R}$ has a transversal homoclinic orbit. In particular, the closure of the interior of the set formed by the diffeomorphisms having zero entropy, is equal to $\overline{\mathcal{MS}}$. In the paper, we have zero entropy if for a C^1 generic diffeomorphism f, a chain transitive set is locally maximal.

THEOREM 2.5. *For C^1 generic $f \in \mathrm{Diff}(M)$, if any chain transitive set Λ of f is locally maximal then it has zero entropy.*

Proof. Let $f \in \mathcal{C} \cap \mathcal{T}$. Since $f \in \mathcal{C} \cap \mathcal{T}$, by Lemmas 2.3 and 2.4 a locally maximal chain transitive set $\Lambda = H_f(p)$. Since $H_f(p)$ a locally maximal homclinic class, the homoclinic class $H_f(p)$ is a relative homoclinic class. Since the homoclinic class $H_f(p)$ is relative homoclinic class, by [2, Theorem 3.1] there is a measure $\mu \in \mathcal{M}_f(H_f(p))$ such that $h_\mu(f) = 0$. $\square$

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